If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

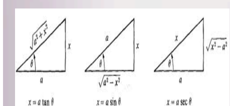
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

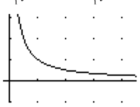
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad+\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from:

the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

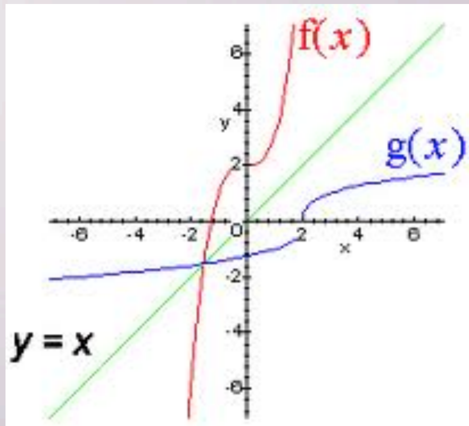
$$\int \frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Inverse Functions



Inverse Functions

If $f(x)$ and $f^{-1}(x)$ are inverse functions:

* $f(x)$ must be one-to-one,
  I.e. The inverse exists when we can get back to an x given a y.
  The horizontal line test may be used.

* If (a,b) is on $f(x)$,then (b,a) is on $f^{-1}(x)$.

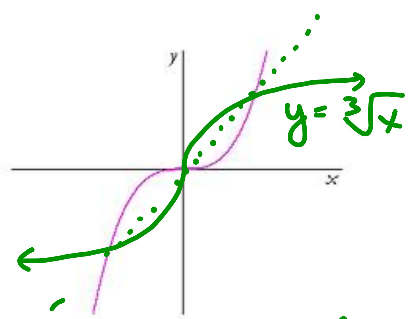
* $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ (f & f^{-1} "undo" each other)

* *The domain of $f(x)$ becomes the range of $f^{-1}(x)$*
* *The range of $f(x)$ becomes the domain of $f^{-1}(x)$*

Note: $f^{-1}(x)$ is not the reciprocal, $\dfrac{1}{f^{-1}(x)}$

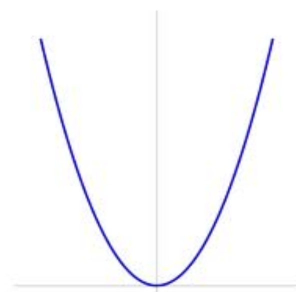
Let's look at two functions:

$f(x) = x^3$



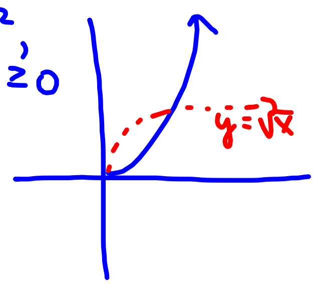
passes horiz. line test
$\Rightarrow$ it has an inverse

$f(x) = x^2$



does not pass horiz. line test (HLT)
$\Rightarrow$ it has no inverse

$y = x^2,$
$x \geq 0$



this passes HLT

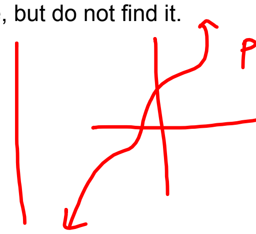
If we don't have a graph, how can we algebraically test if a function has an inverse?

Theorem A If f is strictly monotonic on its domain, then f has an inverse.

monotonic means it's either always nondecreasing or always nonincreasing.
($\Rightarrow$ if $f'(x) \geq 0$ or if $f'(x) \leq 0$)

EX 1 Show that this function has an inverse, but do not find it.

$$f(x) = 3x^7 + 4x^3 + x - 3$$



passes HLT

(note: there is no way to find $f^{-1}(x)$.)

$f'(x) = 21x^6 + 12x^2 + 1 > 0$ for all x.

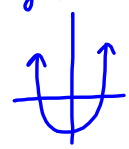
has only pos. coefficients and all even-powered terms

$\Rightarrow f^{-1}(x)$ exists (since $f'(x) > 0$)

EX 2 Explore whether this function has an inverse. If not, can we restrict the domain so it does? If so, find $f^{-1}(x)$.

$$f(x) = x^2 - 4$$

$y = x^2 - 4$



$f'(x) = 2x$ — if $x \geq 0$, $f'(x) \geq 0$; if $x < 0$, $f'(x) < 0$

fn is not monotonic $\Rightarrow f^{-1}(x)$ DNE

however, if we have $f(x) = x^2 - 4$, $x \geq 0$

then $f'(x) \geq 0$ on that domain

$\Rightarrow f^{-1}(x)$ will exist

$f^{-1}(x) = \sqrt{x+4}$

check: $f^{-1}(f(x)) = f^{-1}(x^2 - 4)$

$= \sqrt{(x^2-4)+4}$

$= \sqrt{x^2} = |x|$ but x is restricted $x \geq 0$

$= x$ A ✓

EX 3 Find $f^{-1}(x)$ for this function and check your work.

$f(x) = \quad y = \dfrac{2x-1}{3+5x}$ (review exercise)

(solve for x)

$(3+5x)y = 2x-1$

$3y + 5xy = 2x-1$

$3y + 1 = 2x - 5xy$

$3y+1 = x(2-5y)$

$\dfrac{3y+1}{2-5y} = x \Rightarrow f^{-1}(y) = \dfrac{3y+1}{2-5y}$

$\Rightarrow \boxed{f^{-1}(x) = \dfrac{3x+1}{2-5x}}$

<u>check</u>: $\left(f^{-1}(f(x)) = x \text{ or } f(f^{-1}(x)) = x\right)$

$f^{-1}(f(x)) = f^{-1}\left(\dfrac{2x-1}{3+5x}\right)$

$= \left(\dfrac{3\left(\frac{2x-1}{3+5x}\right)+1}{2-5\left(\frac{2x-1}{3+5x}\right)}\right)\left(\dfrac{3+5x}{3+5x}\right)$

$= \dfrac{3(2x-1)+1(3+5x)}{2(3+5x)-5(2x-1)}$

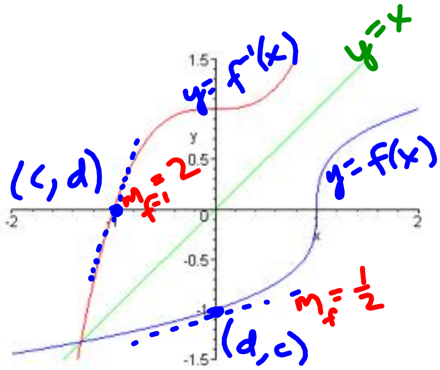
$= \dfrac{6x-\cancel{3}+\cancel{3}+5x}{6+\cancel{10x}-\cancel{10x}+5}$

$= \dfrac{11x}{11} = x$ ✓

<u>note</u>: we should also check (if concerned) that

domain of $f(x)$ = range of $f^{-1}(x)$

and domain of $f^{-1}(x)$ = range of $f(x)$



The graph of $f^{-1}(x)$ is $f(x)$ reflected across the line $y = x$.

Notice the slope at (d,c) and the slope at (c,d).

$$\left(f^{-1}\right)'(d) = \frac{1}{f'(c)}$$

$m_f = \frac{\Delta y_f}{\Delta x}$ $\quad m_{f^{-1}} = \frac{\Delta x_f}{\Delta y_f}$

x, y referring to f(x) graph

in other words slopes are reciprocals at corresponding pts.

Theorem B: Inverse Function Theorem

If f is differentiable, strictly monotonic on an interval and

$f'(x) \neq 0$ at some x on the interval,

then $f^{-1}(x)$ is differentiable at a corresponding point

in the range of f and $\left(f^{-1}\right)'(y) = \dfrac{1}{f'(x)}$.

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

EX 4 Find $(f^{-1})'(2)$ using theorem B. $f(x) = x^5 + 5x - 4$

prove inverse exists

$f'(x) = \underbrace{5x^4}_{\geq 0} + \underbrace{5}_{>0} > 0 \Rightarrow$ $f(x)$ is strictly monotonically increasing

$\Rightarrow f^{-1}(x)$ exists.

slope of $f^{-1}(x) = y$ at $x=2 \equiv$ slope of $y = f(x)$ where $y=2$.

$\Rightarrow 2 = x^5 + 5x - 4$

$6 = x^5 + 5x$

guess $x=1$: $6 \stackrel{?}{=} 1^5 + 5(1)$ yes

$y = f(x)$ goes thru $(1,2)$

$\Rightarrow y = f^{-1}(x)$ goes thru $(2,1)$

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{5(1^4)+5}$$

$$= \boxed{\frac{1}{10}}$$

main pt:

inverse fns $f^{-1}(x)$ exist if $f'(x) \geq 0$

or $f'(x) \leq 0$

(i.e. if f is monotonic,

then $f^{-1}(x)$ exists)