If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

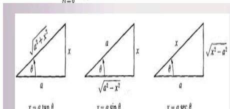
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

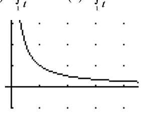
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad+\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

the product rule for differentiation

where it comes from: $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

put into reverse $\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$

$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$

and then rearranged $\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$

Power Series

$$\sum_{n=0}^{\infty} a_n x^n = a_0+a_1x+a_2x^2+a_3x^3+\ldots$$

power series is basically an ∞-degree polynomial

Power Series

Consider a series of functions instead of constants.

Power Series in x $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$ $a_0 = a_0 x^0$

EX 1 When does this power series converge, i.e., for what x-values?

$$\sum_{n=0}^{\infty} a x^n \quad \left\{ \begin{matrix} a \in \Re \\ a \neq 0 \end{matrix} \right\}$$

$$= a \sum_{n=0}^{\infty} x^n$$ (this is just a geometric series)

it converges when $|x| < 1$

$$a \sum_{n=0}^{\infty} x^n = \frac{a}{1-x} = f(x), \quad |x| < 1$$

Theorem

The convergence set for a power series $\sum_{n=0}^{\infty} a_n x^n$ is always an interval of one of these three types.

1) The single point at $x = 0$. $a_0 + a_1(0) + a_2(0) + \ldots = a_0$
2) An interval, $(-R,R)$, $[-R,R]$, $[-R,R)$, or $(-R,R]$ (centered about 0)
3) $(-\infty, \infty)$

The radius of convergence is 0, R, or ∞, respectively.

EX 2 Find the convergence set for $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$.

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Every time we need to find convergence set for a power series, use ART.

ART: $\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^2 (2n)!}{(2n+2)(2n+1)(2n)!} \right|$$

$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1 \Rightarrow \text{converges always}$$

$\to 0$

$\Rightarrow$ this power series converges for all $x \in \mathbb{R}$. interval of convergence is $(-\infty, \infty)$

EX 3 Find the convergence set for $1+x+\frac{x^2}{\sqrt{2}}+\frac{x^3}{\sqrt{3}}+\frac{x^4}{\sqrt{4}}+\ldots$.

n=0 n=1 n=2 n=3 n=4

$$= \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}} \quad \text{or} \quad 1+\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$$

need ART:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right|$$

$$= \lim_{n\to\infty} |x| \left(\frac{\sqrt{n}}{\sqrt{n+1}} \right) = |x| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \to 1$$

$= |x| < 1$ (for convergence)

$-1 < x < 1$ (what about endpts)

test endpts:

$x=1$: $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$ p-series $p = 1/2 < 1$

$\Rightarrow$ diverges

$x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ by AST, this converges

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$\Rightarrow$ the power series $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}}$ converges for $\boxed{-1 \le x < 1}$

c = center value; pivot value; fulcrum

<u>Power Series in *(x-c)*</u>

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

Convergence set :

1) The single point at $x = c$
2) An interval, $(c-R, c+R)$ (may include endpoints)
3) $(-\infty, \infty)$

EX 4 Find the convergence set for $\underset{n=1}{\frac{x-3}{2}} + \underset{n=2}{\frac{(x-3)^2}{2^2}} + \underset{n=3}{\frac{(x-3)^3}{2^3}} + \ldots$.

C = 3 (pivot/center value)

$$= \sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n}$$

use ART:

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n} \right|$$

$$= |x-3| \lim_{n\to\infty} \frac{1}{2} = \underbrace{\frac{1}{2}|x-3| < 1}_{\text{solve for } x}$$

$$|x-3| < 2$$

$$-2 < x-3 < 2$$

$$1 < x < 5$$

test endpts: $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n}$

$x=5$: $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \sum_{n=1}^{\infty} 1$ diverges (by n^{th} term)

$x=1$: $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n$ diverges

⇒ convergence set for the power series is $\boxed{1 < x < 5}$

interval: $(1, 5)$

Conclusion

- Power series are ∞-degree polynomials.
- to find convergence set, use ART.
  (& then test endpts)