If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

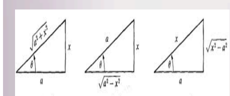
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

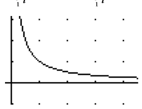
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt\cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Positive Series: Integral Test

Example:

Determine whether the harmonic series

$\sum_{n=1}^{\infty}\frac{1}{n}$ converges or diverges.

Solution:

Using the integral test for convergence:

$$\int_1^{\infty}\frac{dx}{x} = \lim_{a\to\infty}\int_1^{a}\frac{dx}{x} = \lim_{a\to\infty}\ln(a) = \infty$$

$\therefore$ Series diverges

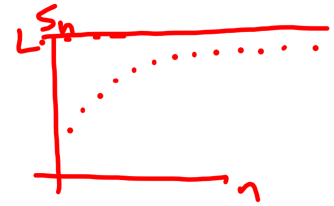
Positive Series: Integral Test

Bounded Sum Test

A series $\sum a_i$ of nonnegative terms converges if and only if its partial sums are bounded above.

creates sequence

remember $S_n = \sum_{i=1}^{n} a_i$ $\quad \{S_n\}$



EX 1 Does $\sum_{k=1}^{\infty} \frac{|\sin k|}{(k+1)!}$ converge?

note: $(k+1)! = (k+1)k(k-1) \cdots 4 \cdot 3 \cdot 2 \cdot 1$

$$= 1 \cdot 2 \cdot 3 \cdot 4 \cdots k(k+1)$$

$$\geq 1 \cdot \underbrace{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2}_{k \text{ times}} = 2^k$$

$$\Rightarrow \frac{1}{(k+1)!} \leq \frac{1}{2^k}$$

$$\Rightarrow \frac{|\sin k|}{(k+1)!} \leq \frac{1}{2^k}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{|\sin k|}{(k+1)!} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = \underbrace{\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k}_{\text{geometric series}} < \infty$$

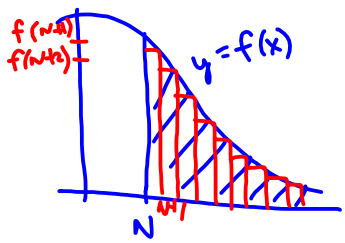
geometric series
$r = \frac{1}{2} < 1$
$\Rightarrow$ converges

$\Rightarrow$ our series converges

Integral Test

If $f(x)$ is continuous①, positive② and nonincreasing③ on $[N,\infty)$ and $a_k = f(k)$ for all positive integers, k, then

$\sum_{n=N}^{\infty} a_n$ converges if and only if $\int_N^{\infty} f(x)dx$ converges.



EX 2 Does $\sum_{k=1}^{\infty} \frac{5k^2}{1+k^3}$ converge or diverge?

(quick: ① not geom. series
② n^{th} term test for divergence
$\lim_{k\to\infty} \frac{5k^2}{1+k^3} = 0 \Rightarrow$ I know nothing)

Try Integral Test: ① positive ✓
② $f(x) = \frac{5x^2}{1+x^3}$ ✓
Cont. everywhere (except at $x=-1$)
③ nonincreasing ✓

$$\int_1^{\infty} \frac{5x^2}{1+x^3}dx = \int_2^{\infty} \frac{5\left(\frac{1}{3}\right)}{u} du$$

$u = 1+x^3$
$du = 3x^2 dx$
$\frac{1}{3}du = x^2 dx$

$x=1,\ u = 1+1^3 = 2$
$x\to\infty,\ u\to\infty$

$$= \frac{5}{3}\lim_{b\to\infty} \int_2^b \frac{1}{u} du$$

$$= \frac{5}{3}\lim_{b\to\infty} \ln|u| \Big|_2^b$$

$$= \frac{5}{3}\left(\lim_{b\to\infty} \cancel{\ln b} - \ln 2\right) \quad \text{diverges}$$

∞

$\rightarrow$ our series diverges

p-series test

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$ is called a p-series. It converges if p > 1 and diverges if p ≤ 1.

Reminder: (in previous lecture on improper integrals)

we proved $\int_1^{\infty} \frac{1}{x^p}\,dx$ $\begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$ (by Integral Test)

Warning: Tell the difference between geometric series and p-series.

$\underbrace{\sum_{q=1}^{\infty} \frac{1}{q^p}}_{\substack{p\text{-series} \\ (q\text{-variable is in base})}}$ converges if $p>1$

$\underbrace{\sum_{q=1}^{\infty} a(r^q)}_{\substack{\text{geometric series} \\ (q\text{-variable is exponent})}}$ converges if $|r|<1$

ex $\sum_{q=1}^{\infty} \frac{1}{q^3}$ vs $\sum_{q=1}^{\infty} \left(\frac{1}{3}\right)^q$

EX 3 Does $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converge or diverge?

(quick: ① not a geom. series
② n^{th} term test:
$\lim_{k \to \infty} \frac{1}{k^3} = 0$
$\Rightarrow$ I know nothing!)

p-series w/ $p = 3 > 1$

$\Rightarrow$ converges

EX 4 Estimate the error made by approximating the series by the sum of the first five terms.

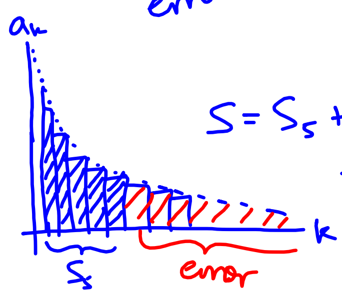
partial sum

$$E_n = \sum_{k=n}^{\infty} \frac{1}{k\sqrt{k}}$$

error

$$S_n = \sum_{k=1}^{n} \frac{1}{k\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{k^{3/2}}$$

$\Rightarrow S = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges (p-series w/ $p = 3/2$)



$S = S_5 + \text{error}$

$\text{error} = S - S_5$

$$\text{error} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} - \sum_{k=1}^{5} \frac{1}{k^{3/2}}$$

$$E_5 = \sum_{k=6}^{\infty} \frac{1}{k^{3/2}}$$

$$E_5 \simeq \int_5^{\infty} \frac{1}{x^{3/2}}\,dx \qquad \text{(worst case error)}$$

$$= \lim_{b \to \infty} \int_5^b x^{-3/2}\,dx$$

$$= \lim_{b \to \infty} -2x^{-1/2} \Big|_5^b = \lim_{b \to \infty} \underbrace{\cancel{\frac{-2}{\sqrt{b}}}}_{0} - \frac{-2}{\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}} \quad \text{error estimate}$$

$$\simeq \boxed{0.894427}$$

in general, if we know what error we can tolerate, then we can determine what n should be to get that error.

we would get $\frac{2}{\sqrt{n}} = \varepsilon$ (ε = error tolerance I want)

solve for n.

Conclusion:

To test for divergence/convergence of a positive infinite series:

① try n^{th} term test for divergence (if n^{th} term $\to$ nonzero as $n \to \infty$, then it diverges; if n^{th} term $\to 0$, we know nothing)

② check if it's

(a) geometric series $\sum_{k=1}^{\infty} a(r^k)$ if $|r|<1$ converges

or (b) p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ if $p>1$ converges

⋮

★1 integral test

★2 partial sum argument

$S_n = \sum_{i=1}^{n} a_i$ if $\lim_{n\to\infty} S_n$ = finite #, then series converges

(particularly useful in collapsing or telescoping sums)