UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Algebraic Topology
May 28, 2021.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points.

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1. Consider the group G given by the presentation

$$G = \langle a, b \mid a^p b^q = 1 \rangle$$

where p, q are given integers.

   (i) Construct a (connected) CW-complex X with a basepoint x such that $\pi_1(X, x) \cong G$.

   (ii) Compute $H_1(X)$ and give conditions on p, q that are equivalent to $H_1(X)$ being torsion-free.

2. Let $p : \tilde{X} \to X$ be a covering map between two connected CW-complexes. If p is null-homotopic, prove that $\tilde{X}$ is contractible. If you use a theorem make sure you state it carefully and include all the hypotheses.

3. Describe a cell structure on the real projective space $\mathbb{R}P^n$. For each cell define its attaching map.

4. Suppose a space Y is obtained from a space X by attaching a single n-cell via an attaching map $f : S^{n-1} \to X$, i.e.

$$Y = X \cup_f e^n$$

Show that $H_i(X) \cong H_i(Y)$ for all i except at most two and give an example where there are two values of i where the two groups are not isomorphic.

5. Let M be a closed connected 5-manifold such that $H_1(M) \cong \mathbb{Z}/3$ and $H_2(M) \cong \mathbb{Z}$. Compute $H_i(M)$ for all i.

6. (i) Define the notion of a chain morphism between two chain complexes and prove that it induces a homomorphism in homology.

(ii) Define the notion of a chain homotopy equivalence.

(iii) Prove that a chain homotopy equivalence induces an isomorphism in homology.