Statistics Prelim Exam
University of Utah
Department of Mathematics

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Read the following instructions before you begin:

- You may attempt all of 10 problems in this exam. On the outside of your exam booklet, indicate which problems you are turning in and want graded.
- Each problem is worth 10 points; 60 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 5080–5090/6010 texts, then you need to carefully state that result.

Exam problems begin here:

1. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t,\theta) = \begin{cases} 0, & \text{if} \quad t < 0 \\ \dfrac{1}{\theta} e^{-t/\theta}, & \text{if} \quad t \geq 0. \end{cases}$$

   Let $\alpha > 0$ and

$$\tau = \frac{1}{\theta^\alpha}.$$

   (a) How big n should be (as a function of α) so that τ has uniformly minimum variance unbiased estimator?

   (b) Find the uniformly minimum variance unbiased estimator for τ (you need to justify that your estimator has the required property).

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t,\theta) = \begin{cases} 0, & \text{if} \quad t \notin [-\theta, \theta] \\ \dfrac{5t^4}{2\theta^5}, & \text{if} \quad -\theta \leq t \leq \theta. \end{cases}$$

   Find the maximum likelihood estimator for θ and compute its bias.

3. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t, \theta) = \begin{cases} 0, & \text{if} \quad t \notin [-\theta, \theta] \\ \dfrac{10t^9}{2\theta^{10}}, & \text{if} \quad -\theta \le t \le \theta. \end{cases}$$

Find a moment estimator for θ.

4. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t\theta) = \frac{1}{2}e^{-|t-\theta|}, \quad -\infty < t < \infty.$$

The parameter θ could be any real number. Find all maximum likelihood estimators for θ.

5. Let $X_1, X_2, \ldots, X_n$ be iid with the same pdf as in question 5. Find the form of the rejection region coming from the generalized likelihood ratio test for $H_0 : \theta = \theta_0$ versus $H_a : \theta \ne \theta_0$.

6. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables that have the uniform distribution on $[0, 1]$. Let $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ be the order statistics. Let $k \ge 1$ be a fixed integer. Show that

$$Y_n = nX_{k,n}$$

converges in distribution and compute the limiting distribution.

7. Let X and Y be independent identically distributed random variables with density function

$$f(t, \theta) = \begin{cases} 0, & \text{if} \quad t < 0 \\ \dfrac{1}{\theta}e^{-t/\theta}, & \text{if} \quad t \ge 0. \end{cases}$$

Compute the density function of

$$Z = \frac{X}{X+Y}.$$

8. Let $X_1, \ldots, X_n \sim N(\mu_1, 1)$ and $Y_1, \ldots, Y_m \sim N(\mu_2, 1)$, where all random variables are independent. Derive the likelihood ratio test for

$$H_0 : \mu_1^2 = \mu_2^2 \quad vs. H_a : \mu_1^2 \ne \mu_2^2.$$

What is the distribution of the test statistic?

9. Suppose a box contains 6 marbles in total. Suppose θ of them are white and $6 - \theta$ of them are black. Test $H_0 : \theta = 2$ against $H_a : \theta \ne 2$ as follows: draw two marbles without replacement and reject H_0 if both marbles are the same color, otherwise do not reject. What is the probability of Type II error for each of the alternatives $\theta = 1$ and $\theta = 3$?

10. Let $X_1, \ldots, X_n$ be iid random variables. The density function of X_i is

$$g(t; \theta_i) = \theta_i t^{-\theta_i - 1} \mathbf{1}\{t \ge 1\}.$$

Here $\theta_i > 0$ is an unknown parameter. We wish to test

$$H_0 : \theta_1 = \theta_2 = \ldots = \theta_n$$

against the alternative that H_0 is not true. Find a test using the generalized likelihood ratio.