Statistics Prelim Exam
University of Utah
Department of Mathematics

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Read the following instructions before you begin:

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problems you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 5080–5090/6010 texts, then you need to carefully state that result.

Exam problems begin here:

1. Let X_1, X_2 be iid with density

$$f(x;\theta) = \begin{cases} 3x^2/\theta^3, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

Find a complete, sufficient statistic for $\theta \in [0, \infty)$ based on X_1 and X_2.

2. Let $X_1, X_2, \ldots, X_n$ be iid random variables with a Beta(θ, θ) distribution, i.e. their density is

$$f(x;\theta) = \frac{\Gamma(2\theta)}{\Gamma(\theta)^2} x^{\theta-1}(1-x)^{\theta-1}, \quad 0 < x < 1.$$

Show that a best critical region for testing $H_0 : \theta = 1$ versus $H_A : \theta = 2$ is

$$\left\{ (x_1, x_2, \ldots, x_n) : c \leq \prod_{i=1}^{n} x_i \right\}.$$

3. A particular gene occurs as one of two alleles (A and a), with allele A having frequency θ in the population. In other words, a random copy of the gene is A with probability θ and a with probability $1-\theta$. A diploid genotype is just two genes put together (in any particular order) so the probability of each diploid genotype is θ^2 for AA, $2\theta(1-\theta)$ for Aa, and $(1-\theta)^2$ for aa. Suppose we test a random sample of people and find that k_1 of them are AA, k_2 of them are Aa, and k_3 of them are aa. Find the MLE of θ.

4. Let X_1, X_2, X_3 be iid and distributed according to $f(x) = 2x$, $0 < x < 1$. Let $X_{(1)} < X_{(2)} < X_{(3)}$ be the order statistics. Compute the probability that the smallest of the X_i exceeds the median of the distribution.

5. Let X be discrete taking values in $\{0,1\}$ and have the pdf $f(x;\theta) = \theta^x(1-\theta)^{1-x}$ for $x \in \{0,1\}$. We test the simple hypothesis $H_0 : \theta = 1/4$ agains the alternative composite hypothesis $H_A : \theta < 1/4$ by taking a random sample of size 10 and rejecting H_0 if and only if the observed values $x_1, \ldots, x_{10}$ satisfy

$$\sum_{i=1}^{10} x_i \leq 1.$$

Find the power function $K(\theta)$ of this test for $0 \leq \theta \leq 1/4$.

6. Consider the density function

$$p(x;\alpha,\lambda) = (2\pi x^3)^{-3/2} \exp\left\{(\alpha\lambda)^{1/2} - \tfrac{1}{2}\log\lambda - \tfrac{1}{2}\alpha x - \tfrac{\lambda}{2}x^{-1}\right\}, \quad 0 < x < \infty.$$

Given $X_1, X_2, \ldots, X_n$ iid from this density find a sufficient statistic (that involves all of the X_i) for the parameters $(\alpha/2, \lambda/2)$.

7. Suppose that $X_1, \ldots, X_n$ form a random sample from a distribution for which the pdf $f(x|\theta_1,\theta_2)$ is as follows:

$$f(x|\theta_1,\theta_2) = \begin{cases} \frac{1}{\theta_2-\theta_1}, & \theta_1 < x < \theta_2 \\ 0, & \text{otherwise} \end{cases}$$

The values of θ_1 and θ_2 are unknown, other than $-\infty < \theta_1 < \theta_2 < \infty$. Find their MLEs.

8. Let $X_1, X_2, \ldots, X_n$ be iid with a Poisson(θ) distribution, so that their common probability mass function is

$$\mathbb{P}(X_i = k) = e^{-\theta}\frac{\theta^k}{k!}, \quad k = 0, 1, 2, \ldots, \ \theta > 0.$$

Find the UMVUE of $e^{-\theta}$.

9. A food processing company packages honey in glass jars. Each jar is supposed to contain 10 fluid ounces of honey. Previous experience suggests that the volume of a randomly selected jar of honey is normally distributed with a known variance of 2 ounces. At a significance level of $\alpha = .05$ derive the likelihood ratio test for testing the null hypothesis $H_0 : \mu = 10$ versus $H_A : \mu \neq 10$.

10. Let $X_1, X_2, \ldots, X_{25}$ denote a random sample of size 25 from a $\mathrm{N}(\theta, 100)$ distribution. Find a uniformly most powerful critical region of size $\alpha = .10$ for testing $H_0 : \theta = 75$ against $H_A : \theta > 75$.