DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
REAL AND COMPLEX ANALYSIS EXAM

January 2, 2008

Instructions:. Do seven problems, at least three from part A and three from part B. List the problems you have done on the front of your blue book.

Part A.

1. Let (X, μ) be a measure space and $f \geq 0$ an integrable function on X. Prove that for $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\mu(A) < \delta$, then $\int_A f < \epsilon$.

2. Let K be a continuous function on the square $[0,1] \times [0,1]$, and let $[0,1]$ be endowed with the Lebesgue measure. For $f \in L^2([0,1])$ define $Tf(x) = \int_0^1 K(x,y)f(y)\,dy$. Show that $Tf \in L^2([0,1])$ and that $T : L^2 \to L^2$ is a bounded operator with $\|T\| \leq \|K\|_\infty$.

3. Let H be a Hilbert space, and let $T : H \to H$ be a self adjoint linear operator, with finite dimensional range $V = T(H)$. Prove that T is a compact operator, and that $V = K^\perp$, where K is the kernel of T.

4. Let $f \in L^1([-\pi, \pi])$, and let c_n be the nth Fourier coefficient of f:

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \quad n \in \mathbf{Z}$$

Prove that $c_n \to 0$ as $|n| \to \infty$.

5. Let ℓ^∞ denote the Banach space of bounded real sequences with norm $\|x\|_\infty = \sup_i |x_i|$, and ℓ^1 the Banach space of real sequences with norm $\|x\|_1 = \sum_i |x_i| < \infty$. For $x \in \ell^\infty$ and $y \in \ell^1$ define $Tx(y) = \sum_i x_i y_i$. Prove that $T : \ell^\infty \to \ell^{1*}$ is a bounded operator that is one to one and onto. (Here ℓ^{1*} is the dual space of ℓ^1.)

Part B.

6. Let Q be a square in $\mathbf{C}$ and $f : Q \to \mathbf{C}$ a continuous map that is holomorphic on the interior of Q. Also assume that if $z \in \partial Q$ then $|f(z)| = 1$. Show that f extends to a holomorphic map on a neighborhood of Q.

7. Show that the function $f(z) = \frac{\cos z}{z^2}$ is the complex derivative of a holomorphic function F on $\mathbf{C} \backslash \{0\}$. Write down a Laurent series for F.

8. Let Δ be the open unit disk and γ the unit circle in $\mathbf{C}$ and $\phi : \gamma \to \mathbf{C}$ a continuous function. Let g be a meromorphic function on $\mathbf{C}$ that has a single simple pole at 0. Define a function $f : \Delta \to \mathbf{C}$ by the formula

$$f(z) = \int_\gamma \phi(w)g(w-z)dw.$$

Show that f is holomorphic.

9. Let f be a non-constant meromorphic function on $\mathbf{C}$. Show that either there exists a sequence $z_n \to \infty$ with $f(z_n) \to 0$ as $n \to \infty$ or there is a $z \in \mathbf{C}$ such $f(z) = 0$.

10. Let Ω be an open subset of $\mathbf{C}$ and $f : \Omega \to \mathbf{C}$ a holomorphic function. Let z be a point in Ω and assume that Ω contains a disk D centered at z of radius R. Also assume that $f(D)$ is contained in a disk D' centered at $f(z)$ of radius r. Show that $|f'(z)| \leq r/R$.