Probability qualifying exam–January, 2015.

You need to get at least 50 points to pass. The value of each question is 10 points.

(1) $X_1, X_2, \ldots, X_n$ be independent identically distributed positive random variables. Show that

$$\frac{1}{n} \max_{1 \le k \le n} X_k \to 0 \quad \text{a.s.}$$

if and only if $EX_1 < \infty$.

(2) Let $X \ge 0$ and $E|X|^p < \infty$ with some $p > 0$. Show that

$$\lim_{t \to \infty} t^p P\{X > t\} = 0.$$

(3) Let X and Y be independent random variables. Show that if the distribution of X is absolutely continuous with respect to the Lebesgue measure, then the distribution of $X + Y$ is also absolutely continuous with respect to the Lebesgue measure.

(4) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $E \log(1 + |X_1|) < \infty$. Show that

$$\sum_{i=1}^{\infty} \prod_{j=1}^{i} X_j \quad \text{is finite with probability one if and only if } E \log |X_1| < 0.$$

(5) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with distribution function $F(x) = P\{X_1 \le x\}$. Show that

$$\sup_{-\infty < x < \infty} |F_n(x) - F(x)| \to 0 \quad \text{a.s.}$$

as $n \to \infty$, where

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I\{X_i \le x\}.$$

(6) Let $X_1, X_2, \ldots, X_n$ be independent identically distributed exponential(1) random variables. Let $k \ge 1$. Show that

$$P\{X_n > \phi_\varepsilon(n) \ \text{ i.o.}\} = \begin{cases} 0, & \text{if } \ \varepsilon > 0 \\ 1, & \text{if } \ \varepsilon \le 0, \end{cases}$$

where $\phi_\varepsilon(n) = \log n + \log_{(2)}(n) + \ldots + (1+\varepsilon) \log_{(k)}(n)$, and $\log_{(1)}(n) = \log n$, $\log_{(k)}(n) = \log(\log_{(k-1)}(n))$.

(7) Let $X_1, X_2, \ldots$ be independent identically distributed positive random variables with $EX_1 = \mu$. Define

$$N(t) = \min\{k : \sum_{i=1}^{k} X_i > t\}.$$

Show that

$$\lim_{t \to \infty} \frac{1}{t} N(t) = \frac{1}{\mu} \quad \text{a.s.}$$

(8) Let X and Y be independent standard normal random variables. Compute the density function of $Z = X/Y$.

(9) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables uniform on $[0,1]$. Show that for all $\alpha > -1$ there are numerical sequences a_n and b_n such that

$$Y_n = \frac{\sum_{1\le k\le n} k^\alpha X_k - a_n}{b_n}$$

converges in distribution to a standard normal random variable.

(10) Compute

$$\lim_{n\to\infty} e^{-n} \sum_{i=0}^{n+\sqrt{n}} \frac{n^i}{i!}.$$