Probability Prelim Exam

August 2016

Instructions (Read before you begin)

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Construct an example of a countable family $\{\mathscr{F}_i\}_{i=1}^\infty$ of σ-algebras such that $\sigma(\cup_{i=1}^\infty \mathscr{F}_i) \neq \cup_{i=1}^\infty \mathscr{F}_i$. Can you do this so that $\mathscr{F}_i \subset \mathscr{F}_{i+1}$ for all $i \geqslant 1$? Justify your reasoning.

2. Choose and fix two integers $N \geqslant 1$ and $x \in \{0, 1, \ldots, N\}$. Let $X_0 = x$ and suppose that, conditionally, X_{n+1} has a Binomial$(N\,, X_n/N)$ distribution given $X_1, \ldots, X_n$. More precisely, for all $n \geqslant 0$ and $k = 0, \ldots, N$,

$$\mathrm{P}\left(X_{n+1} = k \mid \mathscr{F}_n\right) = \binom{N}{k}\left(\frac{X_n}{N}\right)^k\left(1 - \frac{X_n}{N}\right)^{N-k},$$

   where $\mathscr{F}_n = \sigma(\{X_1\,, \ldots, X_n\})$ for all $n \geqslant 1$. Prove that:

   (a) $X_\infty = \lim_{n\to\infty} X_n$ exists a.s. and in $L^p(\mathrm{P})$ for all $1 \leqslant p < \infty$.
   (b) Prove that $\mathrm{E}(X_\infty) = x$ and $\mathrm{E}(X_\infty^2) = Nx$.
   (c) Compute the distribution of X_∞.

3. Let $n \geqslant 1$ be a fixed integer and recall that there are $n!$ permutations of $[n] := \{1\,, \ldots, n\}$. Let $\sigma := \{\sigma(1)\,, \ldots, \sigma(n)\}$ denote an arbitrary permutation of $[n]$, and choose and fix some integer $j \in [n]$. We say that σ *fixes* j if $\sigma(j) = j$. Now suppose that σ is selected at random, all permutations of $[n]$ equally likely.

   (a) What is the probability that σ fixed j?
   (b) What is the expectation of the number of integers in $[n]$ that are fixed by σ?

4. Construct two uncorrelated random variables that are not independent.

5. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables, and let $p > 0$ be fixed. Prove that the following are equivalent:

   (a) $\mathrm{E}(|X_1|^p) < \infty$;
   (b) $n^{-1/p} X_n \to 0$ almost surely; and
   (c) $n^{-1/p} \max_{1 \leqslant j \leqslant n} |X_j| \to 0$ almost surely.

6. Suppose $\{X_n\}_{n=1}^{\infty}$ are i.i.d. exponential random variables with mean one; that is, $\mathrm{P}\{X_n > x\} = \mathrm{e}^{-x}$ for all $x > 0$. Prove that

$$\mathrm{P}\left\{ \limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \right\} = 1.$$

7. Suppose $X_1, X_2, \ldots$ are i.i.d. strictly positive random variables. Compute for all $n, m, k \geqslant 1$,

$$\mathrm{E}\left(\frac{X_1^k + \cdots + X_m^k}{X_1^k + \cdots + X_n^k} \right)$$

   You might wish to start by calculating $\mathrm{E}(X_1^k \mid X_1^k + \cdots + X_n^k)$.

8. Let X and Y be two positive random variables. We say that X is *stochastically dominated by* Y if $\mathrm{P}\{X > a\} \leqslant \mathrm{P}\{Y > a\}$ for all $a > 0$. Prove that, in this case, $\mathrm{E}(X^k) \leqslant \mathrm{E}(Y^k)$ for all $k \geqslant 1$.

9. Suppose $X_1, N_1, X_2, N_2, \ldots$ are independent random variables such that the X's are Bernoulli($1/2$)—that is, $\mathrm{P}\{X_i = 0\} = \mathrm{P}\{X_i = 1\} = 1/2$ for all $i \geqslant 1$—and each N_n is Poisson(n); that is,

$$\mathrm{P}\{N_n = k\} = \frac{n^k \mathrm{e}^{-n}}{k!} \qquad \text{for all integers } n \geqslant 1 \text{ and } k \geqslant 0.$$

   Find non-random sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that, as $n \to \infty$,

$$\frac{\sum_{i=1}^{N_n} X_i - a_n}{b_n} \Rightarrow \mathrm{N}(0\,, 1).$$

10. Suppose $\mathrm{P}\{X_1 = 0\} = \mathrm{P}\{X_1 = 1\} = 1/2$ and for all integers $n \geqslant 1$,

$$\mathrm{P}\left(X_{n+1} = 1 \mid \mathscr{F}_n \right) = 1 - \mathrm{P}\left(X_{n+1} = 0 \mid \mathscr{F}_n \right) = \frac{S_n}{n},$$

   where $S_n := X_1 + \cdots + X_n$ and $\mathscr{F}_n := \sigma(\{X_1, \ldots, X_n\})$. Prove that $n^{-1} S_n = X_1$ almost surely for all $n \geqslant 1$.