Preliminary Exam, Numerical Analysis, August 2008

Instructions: This exam is closed books and notes, and no electronic devices are allowed. The allotted time is three hours and you need to work on any three out of questions 1-4 and any two out of questions 5-7. All questions have equal weight and a score of 75 % is considered a pass. Indicate clearly the work that you wish to be graded.

-1- (QR factorization.) Show how to compute the QR factorization of a real $m \times n$ matrix A (with $m \geq n$) using Householder reflections, and describe how to use the QR factorization to solve the Least Squares problem

$$\|Ax - b\|_2 = \min . \tag{1}$$

-2- (Gaussian Quadrature.) Let w be a positive weight function. Explain how to pick the knots x_i and the weights w_i, $i = 1, \ldots, n$ such that the integration formula

$$\int_a^b w(x) f(x) \mathrm{d}x = \sum_{i=1}^{n} w_i f(x_i) \tag{2}$$

is exact for polynomials f of degree as high as possible, and show why this works. How high a polynomial degree is possible?

-3- (Interpolation error.) Let $I = [a, b]$ and $f \in C^\infty[a, b]$. Let x_i, $i = 0, \ldots, n$ be $n + 1$ distinct points in I. Let $p(x)$ be the unique polynomial of degree at most n satisfying $p(x_i) = f(x_i)$, $i = 0, \ldots, n$. Given x in $[a, b]$, show that there exists some point $\xi \in [a, b]$ such that

$$f(x) - p(x) = \frac{\prod_{i=0}^{n}(x - x_i)}{(n + 1)!} f^{(n+1)}(\xi). \tag{3}$$

-4- (Singular Value Decomposition.) Let A be an $m \times n$ matrix. You may assume that $m \geq n$. Define what is meant by the *singular value decomposition of* A and show that every $m \times n$ matrix has one.

-5- (Absolute Stability.) Consider solving the initial value problem

$$y' = f(x, y), \qquad y(a) = y_0 \tag{4}$$

by the linear multistep method (LMM)

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{5}$$

where we use the standard notation

$$x_n = a + nh, \qquad y_n \approx y(x_n), \qquad f_n = f(x_n, y_n). \tag{6}$$

Define what it means that the LMM is *explicit.* Define its *region of absolute stability.* Show that an explicit linear multistep method cannot have an infinite region of absolute stability.

-6- (Convergence.) Consider the one-dimensional heat equation initial boundary value problem

$$u_t = u_{xx}, \qquad x \in [0,1], \qquad t \geq 0, \qquad u(x,0) = f(x), \qquad u(0,t) = u(1,t) = 0 \tag{7}$$

and the discretization

$$x_m = mh, \qquad t_n = nk, \quad \text{and} \quad U_m^n \approx u(x_m, t_n) \tag{8}$$

where

$$U_m^{n+1} = U_m^n + r\left(U_{m-1}^n - 2U_m^n + U_{m+1}^n\right) \tag{9}$$

and $r = k/h^2$ is the grid constant. Define what it means that this method is convergent, say for what values of r it is convergent, and show that your statement is correct.

-7- (Local Truncation Error.) Consider the wave equation

$$u_{tt} = c^2 u_{xx} \tag{10}$$

and its discretization

$$x_m = mh, \qquad t_n = nk, \qquad U_m^n \approx u(x_m, t_n) \tag{11}$$

and

$$U_m^{n+1} - 2U_m^n + U_m^{n-1} = r(U_{m+1}^n - 2U_m^n + U_{m-1}^n) \tag{12}$$

where $r = c^2k^2/h^2$ is the grid constant. For the purposes of this problem, ignore the issues of initial and boundary conditions. Compute the local truncation error of (12) and show that there is a value of r for which the local truncation error is exactly zero. Give a physical interpretation of this fact.