Preliminary Examination, Numerical Analysis, January 2017

Instructions: This exam is closed books and notes. The time allowed is three hours and you need to work on any three out of questions 1-5 and any two out of questions 6-8. All questions have equal weights and the passing score will be determined after all the exams are graded. Indicate clearly the work that you wish to be graded.

Note: In problems 6-8, the notations $k = \Delta t$ and $h = \Delta x$ are used.

1. Singular Value Decomposition (SVD):

a) Prove the following statement:

Singular Value Decomposition: Any matrix $A \in \mathbb{C}^{m\times n}$ can be factored as $A = U\Sigma V^*$, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{R}^{m\times n}$ is a rectangular matrix whose only nonzero entries are non-negative entries on its diagonal.

b) Relate the matrices U, V, and Σ to the four fundamental subspaces associated with A, that is, the range and null spaces of A and A^T.

2. Linear Least Squares:

The Linear Least Squares problem for an $m \times n$ real matrix A and $b \in \mathbb{R}^m$ is the problem:

$$\text{Find} \quad \mathrm{x} \in \mathbb{R}^{\mathrm{n}} \text{ such that } \|\mathrm{Ax} - \mathrm{b}\|_2 \text{ is minimized.}$$

a) Suppose that you have data $\{(t_j, y_j)\}$, $j = 1, 2, \ldots, m$ that you wish to approximate by an expansion

$$p(t) \;=\; \sum_{k=1}^{n} x_k \phi_k(t).$$

Here, the functions $\phi_k(t)$ are given functions. Which norm on the difference between the approximation function p and the data gives rise to a linear least squares problem for the unknown expansion coefficients x_k? What is the matrix A in this case, and what is the vector b?

b) Suppose that A is a real $m \times n$ matrix of full rank and let $b \in \mathbb{R}^m$. What are the 'normal equations' for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the QR factorization of A and how can it be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.

3. Sensitivity:

a) Suppose that A is an $n \times n$ nonsingular real matrix. Analyze the sensitivity of solutions of the system $A\mathbf{x} = \mathbf{b}$ to perturbations in $\mathbf{b}$. What quantity related to A characterizes this sensitivity?

b) Suppose $\tilde{\mathbf{x}}$ is an approximate solution to the linear system $A\mathbf{x} = \mathbf{b}$, where A is an $n \times n$ nonsingular real matrix. The residual is the vector $\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}}$. Derive inequalities relating the residual $\mathbf{r}$ to the error $\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$.

4. Interpolation and Integration:

a) Consider $n + 1$ distinct points $x_0 < x_1 < ... < x_n$ in the interval $[a, b]$. Let $f(x)$ be a smooth function defined on $[a, b]$. Show that there is a unique polynomial $p(x)$ of degree n which interpolates f at all of the points x_j. Derive the formula for the interpolation error at an arbitrary point x in the interval $[a, b]$:

$$f(x) - p(x) \equiv E(x) = \frac{1}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n) f^{n+1}(\eta).$$

for some $\eta \in [a, b]$.

b) Consider the problem of approximating the integral $I(f) = \int_a^b f(x)dx$ by a formula of the type $I_n(f) = \sum_{j=1}^{n} a_j f(x_j)$ where $x_1, x_2, ..., x_n$ are distinct points in the interval (a, b). Derive formulas for $a_j, j = 1, ..., n$ so the $I_n(f) = I(f)$ when f is any polynomial of degree less than or equal to n.

c) For the same approximate integration problem as in (b), explain how to choose the points $x_1, x_2, ..., x_n$ and coefficients $a_j, j = 1, ..., n$, so that $I_n(f) = I(f)$ for all polynomials of degree less than or equal to $2n - 1$? Prove that your proposed choice does give the exact integral for these polynomials.

5. Iterative Methods:

Consider the fixed-point iteration

$$\mathbf{u}^{(k+1)} = T\mathbf{u}^{(k)} + \mathbf{c}$$

for finding a solution of the problem

$$\mathbf{u} = T\mathbf{u} + \mathbf{c},$$

where T is an $m \times m$ real matrix and $\mathbf{c}$ is a real m-vector.

a) Show that the fixed point iteration will converge for an arbitrary initial guess $\mathbf{u}^{(0)}$ if and only if the spectral radius of T, $\rho(T)$, is less than 1.

b) Consider the boundary value problem

$$-u''(x) = f(x), \quad \text{for } 0 \ \leq \ \text{x} \ \leq \ 1$$

with $u(0) = u(1) = 0$, and the following discretization of it:

$$-U_{j-1} + 2\ U_j - U_{j+1} = F_j,$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$ and $F_j \equiv h^2\ f(jh)$.

Show that the Jacobi iterative method will converge for this problem for any choice of initial guess. Express the speed of convergence as a function of the discretization stepsize h. How does the number of iterations required to reduce the initial error by a factor δ depend on h? In practice, would you use this method to solve the given problem? If so, explain why this is a good idea? If not, how would you solve it in practice?

6. Elliptic Problems:

For the one dimensional Poisson problem for $v(x)$

$$-v''(x) + \alpha v(x) = f(x),$$

where $\alpha \geq 0$ is constant, along with Dirichlet boundary conditions in the interval [0,1], consider the scheme

$$\Delta_h U_j \equiv \frac{1}{h^2}\Big(- U_{j-1} + 2\ U_j - U_{j+1}\Big) = f_j$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$, $f_j \equiv f(jh)$, and $U_0 = U_N = 0$. The approximate solution satisfies a linear system $AU = b$, where $U = (U_1, U_2, ..., U_{N-1})^T$ and $b = h^2(f_1, f_2, ..., f_{N-1})^T$.

a) State and prove the maximum principle for any grid function $V = \{V_j\}$ with values for $j = 0, 1, ...N$, that satisfies $\Delta_h V_j \geq 0$.

b) Derive the matrix A and show that it is symmetric and positive definite.

c) Use the maximum principle to show that the global error $e_j = v(x_j) - U_j$ satisfies $\|e\|_\infty = O(h^2)$ as the space step $h \to 0$.

7. Numerical Methods for ODEs:

Consider the Linear Multistep Method

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}kf_{n+2}$$

for solving an initial value problem $y' = f(y, x)$, $y(0) = \eta$. You may assume that f is Lipschitz continuous with respect to y uniformly for all x.

a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

b) Sketch a graph of the solution to the following initial value problem.

$$y' = -10^8[y - \cos(x)] - \sin(x), \quad y(0) = 2.$$

Would it be more reasonable to use this method or the forward Euler method for this problem? What would you consider in choosing a timestep k for each of the methods? Justify your answer.

8. Heat Equation Stability:

Consider the variable coefficient diffusion equation

$$v_t = (\beta(x)v_x)_x, \qquad 0 < x < 1, \ t > 0$$

with Dirichlet boundary conditions

$$v(0,t) \ = \ 0, \qquad v(1,t) \ = \ 0$$

and initial data $v(x, 0) = f(x)$. Assume that $\beta(x) \geq \beta_0 > 0$, and that $\beta(x)$ is smooth. Let $\beta_{j+1/2} = \beta(x_{j+1/2})$. A scheme for this problem is:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{h^2} \left\{ \beta_{j-1/2}u_{j-1}^{n+1} - (\beta_{j-1/2} + \beta_{j+1/2})u_j^{n+1} + \beta_{j+1/2}u_{j+1}^{n+1} \right\}.$$

Analyze the 2-norm stability of this scheme for solving this initial boundary value problem.

DO NOT NEGLECT THE FACT THAT THE PROBLEM HAS VARIABLE COEFFICIENTS AND THAT THERE ARE BOUNDARY CONDITIONS AT 0 AND 1!

Fact 1: A real symmetric $n \times n$ matrix A can be diagonalized by an orthogonal similarity transformation, and A's eigenvalues are real.

Fact 2: The $(N-1) \times (N-1)$ matrix M defined by

```
[ -2  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -2 ]
```

has eigenvalues $\mu_l \;=\; -4\sin^2(\frac{\pi l}{2N})$, $l = 1, 2, \ldots, N-1$.

Fact 3: The $(N+1) \times (N+1)$ matrix:

```
[ -1  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -1 ]
```

has eigenvalues $\mu_l = -4\sin^2\left(\frac{\pi l}{2(N+1)}\right), \quad l = 0, 1, \ldots, N.$

Fact 4: For a real $n \times n$ matrix A, the Rayleigh quotient of a vector $x \in R^n$ is the scalar

$$r(x) = \frac{x^T A x}{x^T x}.$$

The gradient of $r(x)$ is

$$\nabla r(x) = \frac{2}{x^T x}(Ax - r(x)x).$$

If x is an eigenvector of A then $r(x)$ is the corresponding eigenvalue and $\nabla r(x) = 0$.