DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

January 2008

Instructions: Do four (4) problems from section A and four (4) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first four answered will be scored. For a pass, three problems from each group have to be solved entirely.

A. Answer four of the following questions. Each question is worth ten points. All manifolds are smooth.

1. Let $M = \{z \in \mathbf{C}^n = \mathbf{R}^{2n} : z_1^2 + \cdots z_n^2 = 1\}$. Prove that M is a regular submanifold of $\mathbf{R}^{2n}$, and find it's dimension.

2. Let G be an n-dimensional Lie group, and TG it's tangent bundle. Prove that TG is diffeomorphic to $G \times \mathbf{R}^n$.

3. Let M be a compact orientable manifold with boundary. Use differential forms to show that there is no smooth retraction $r : M \to \partial M$.

4. Let M be a manifold. Prove that the tangent bundle TM and the cotangent bundle TM^* are isomorphic (as smooth vector bundles).

5. Let M be a Riemannian manifold. Define $d : M \times M \to \mathbf{R}$ by

$$d(x, y) = \inf \int_0^1 |\gamma'(t)| dt$$

where the infimum is taken over piecewise C^1 paths γ in M with $\gamma(0) = x$ and $\gamma(1) = y$. Prove that d is a metric on M.

6. Let M be an orientable manifold of dimension n. Prove that the top dimensional de Rham cohomology with compact supports $H_c^n(m)$ is nontrivial.

7. Let $T = S^1 \times S^1 \subset \mathbf{C} \times \mathbf{C}$ be the torus. Let X be the vector field on T defined by $X(z_1, z_2) = (iz_1, aiz_2)$, where $a \in \mathbf{R}$. Compute the flow $\Phi(t, p)$, $p = (p_1, p_2)$ of X. For a rational, show that $t \to \Phi(t, p)$ is periodic.

B. Answer four of the following questions. Each question is worth ten points.

1. Compute $H_i(S^n - X)$ when X is a subspace of S^n homeomorphic to $S^p \vee S^q \vee S^r$ with $p, q, r \in \{1, 2, \cdots, n-1\}$ and $p \neq q \neq r \neq p$.

2. Let S_g be the closed, orientable surface of genus g. Calculate $\pi_2(S_g)$ for all g.

3. Prove that the map $h : S^3 \to \mathbb{C}P^1$ given by $h(x, y) = [x : y]$ is a fiber bundle. Here, S^3 is the unit sphere $|x|^2 + |y|^2 = 1$ in $\mathbb{C}^2$.

4. Let M be an open 3-manifold such that every compact set of M is contai ned in a solid torus.

   (a) Show that $\pi_1(M)$ is abelian.

   (b) Assume that $\pi_1(M)$ is finitely generated. Show that $\pi_1(M)$ is trivial or $\mathbb{Z}$.

5. Define the higher homotopy groups, π_i, for $i \geq 2$ and prove that they are abelian.

6. Let $p : \tilde{X} \to X$ be a 2-sheeted covering map between cell complexes such that the cell structure on $\tilde{X}$ is obtained by lifting the cell structure on X.

   (a) Construct a short exact sequence of cellular chain complexes with coefficients in $\mathbb{Z}/2\mathbb{Z}$

   $$0 \to C_n(X; \mathbb{Z}/2\mathbb{Z}) \to C_n(\tilde{X}, \mathbb{Z}/2\mathbb{Z}) \to C_n(X, \mathbb{Z}/2\mathbb{Z}) \to 0$$

   (i.e. give the maps explicitly and prove that the sequence is exact).

   (b) Write down the induced long exact sequence of homology groups (the transfer sequence).

   (c) Prove that if $\tilde{X}$ is acyclic over $\mathbb{Z}/2\mathbb{Z}$ (i.e. $H_i(\tilde{X}; \mathbb{Z}/2\mathbb{Z}) = 0$ for $i > 0$ and $H_0(\tilde{X}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$) then $H_i(X; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for all $i \geq 0$.

7. Let Σ be a closed oriented surface of genus g and suppose that $X \subset \Sigma$ is a graph which is a retract of Σ. Prove that

   $$rank H_1(X) \leq g$$

8. Let Σ_g denote the closed oriented surface of genus g. Show that every map $f : \Sigma_2 \to \Sigma_3$ has degree 0.