DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

August, 2017

Instructions: Provide solutions for as many problems as you can in the time allowed. Divide your efforts on both sections, A and B, as you'll have to pass both parts to pass the qualifying exam as a whole. Cite the theorems that you use.

Section A.

1. Let $\mathbf{P^n}(\mathbb{R})$ be n-dimensional real projective space. Use an explicit collection of charts to prove that $\mathbf{P^n}(\mathbb{R})$ is a smooth manifold.

2. Prove that there is no immersion of the n-sphere into $\mathbb{R}^n$.

3. Let M be the smooth submanifold of $\mathbb{R}^3$ that is the solution set of the equation $x^2 + y^2 - z^2 = 1$. Let N be the smooth submanifold of $\mathbb{R}^3$ that is the solution set of the equation $x^2 + y^2 + z^2 = 1$. Prove that M and N do not intersect transversally in $\mathbb{R}^3$.

4. Let M and N be smooth manifolds, and let M be compact. Suppose $F : M \times [0,1] \to N$ is a smooth function, and for any $t \in [0,1]$ let $F_t : M \to N$ be the function defined by $F_t(p) = F(p,t)$. If F_0 is an immersion, prove that there is some $\varepsilon \in (0,1]$ such that F_t is an immersion for all $t \in [0,\varepsilon)$.

5. Let G be a Lie group, and let H be the connected component of G that contains the identity. Prove that H is a Lie subgroup of G, and that H is a normal subgroup of G. (Do not use the theorem that closed subgroups of Lie groups are Lie groups.)

6. Let Σ be a closed, smooth surface. Show that if Σ admits a nonvanishing vector field, then the Euler characteristic of Σ equals 0.

Section B.

7. Prove that if X and Y are path connected spaces, then $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

8. Let Σ_g be a closed, orientable surface of genus $g \geq 1$. Prove that $\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i]\rangle$.

9. Show that $\pi_1(\Sigma_2)$ contains $\pi_1(\Sigma_3)$ as a normal subgroup.

10. Let D^k be the closed k-dimensional disk. Use that $D^k/\partial D^k = S^k$ to find $H_n(S^k;\mathbb{Z})$ for all n.

11. Suppose $f : S^k \to S^k$ is continuous and not surjective. Prove that $f_* : H_k(S^k;\mathbb{Z}) \to H_k(S^k;\mathbb{Z})$ is the zero homomorphism.

12. Suppose M is a closed, orientable, simply connected, smooth manifold of dimension 3. Let S_3 be the symmetric group on 3 letters, and suppose S_3 acts on M freely, by orientation preserving diffeomorphisms. Let $N = S_3 \backslash M$, and find $H_n(N;\mathbb{Z})$ and $H^n(N;\mathbb{Z})$ for all n.