Allow yourself 90 minutes for Part A, and 90 minutes for Part B.

A. Answer all of the following questions.

1. Prove that S^2 is not a Lie group.

2. Give an example of a foliation on a compact manifold M such that there are infinitely many leaves, and each leaf is dense M.

3. For any $p \in \mathbb{R}^3$, let K_p be the kernel of the form $2dz - 3ydx$ taken at the point p. Then K_p defines a plane field on the manifold $\mathbb{R}^3$. Is the plane field integrable? Why?

4. Show that the Lie algebra of $\mathrm{SL}_3(\mathbb{R})$ is the space of 3×3 matrices whose trace equals 0.

5. Suppose $f : M \to N$ is a smooth function of smooth manifolds. Suppose Q is an embedded submanifold of N, and that f is transverse to Q. Prove that if $p \in f^{-1}(Q)$, then

$$T_p(f^{-1}(Q)) = (D_p f)^{-1}[T_{f(p)}Q]$$

(The above line says that the tangent space for $f^{-1}(Q)$ at p is the inverse image of the tangent space of Q at $f(p)$ under the differential of f at p.)

6. Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then prove there is a Lie subgroup $H \leq G$ whose Lie algebra is $\mathfrak{h}$.

B. Answer all of the following questions.

7. Give an example of an irregular (i.e. not normal) covering space of the Klein bottle (with a proof).

8. Let X be a compact topological space and $\widetilde{X}$ its universal cover. Show that $\pi_1(X)$ is finite if and only if $\widetilde{X}$ is compact.

9. Prove that $H_i(S^2 \vee S^4, \mathbb{Z})$ and $H_i(\mathbb{CP}^2, \mathbb{Z})$ are isomorphic for all i, but that $S^2 \vee S^4$ is not homotopy equivalent to $\mathbb{CP}^2$.

10. Let Σ be a closed oriented surface of genus g and suppose that $X \subset \Sigma$ is a graph that is a retract of Σ. Prove that:

$$\mathrm{rank}(H_1(X)) \leq g$$

11. Let M be a closed, orientable manifold of dimension $2k$ and assume that $H_{k-1}(M, \mathbb{Z})$ is torsion-free. Show that $H_k(M, \mathbb{Z})$ is torsion free.

12. Let G be a group of homeomorphisms acting freely on the n-sphere S^n with n even. Prove that G has at most 2 elements.