# PhD Preliminary Qualifying Examination

# Applied Mathematics

Wednesday January 6 2016

Instructions: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

**Part A.**

1. Let $X$ be a Banach space, let $z \in X$ with $z \neq 0$, and let $f \in X'$ with $f \neq 0$. Consider the linear operator $T : X \to X$ defined for $x \in X$ by
$$Tx = f(x)z.$$
   (a) Briefly explain why $T$ is compact.
   (b) Use **Neumann series** to give a condition on $f$ and $z$ that ensures the equation $x + Tx = y$ admits a unique solution $x$ for all $y \in X$.
   (c) Use the **Fredholm alternative** to find a condition on $f$ and $z$ that ensures the equation $x + Tx = y$ admits a unique solution $x$ for all $y \in X$.

2. Let $X$ and $Y$ be normed vector spaces. Show that a linear operator $T : X \to Y$ is compact if and only if for every sequence $(x_n)$ with $\|x_n\| \leq 1$, the sequence $(Tx_n)$ has a convergent subsequence.

3. Let $T$ be a bounded linear operator on a complex Hilbert space $H$. Show that the operator $I + T^*T$ is injective.

4. Let $Y$ be a subspace of a Hilbert space $H$. Show that $Y$ is closed if and only if $Y = Y^{\perp\perp}$.

5. Let $X$ be a Banach space. Let $(f_n)$ be a sequence in $X'$.
   (a) Show that $f_n \to f$ weakly in $X'$ implies that for all $x \in X$, we have $f_n(x) \to f(x)$.
   (b) If there is an $f \in X'$ such that $f_n(x) \to f(x)$ for all $x \in X$, does $f_n \to f$ weakly in $X'$? Explain your answer.

**Part B.**

1. Find the sum

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos 99x\,.$$

2. Prove the fundamental theorem of algebra:
*Every polynomial of degree $n(\geq 1)$ has exactly $n$ zeros (each zero being counted according to its multiplicity).*

3. Evaluate the integrals

$$(a) \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x} dx,$$

$$(b) \qquad \int_{0}^{\infty} \frac{x^\alpha}{x+1} dx.$$

4. Show that Fourier transform

$$f(x) \to F(\mu) = \int_{-\infty}^{\infty} f(x)\, e^{i\mu x}\, dx$$

preserves the inner product and the $L_2$ norm (Plancherel's equations):

$$\int_{-\infty}^{\infty} \bar{f}(x)\, g(x)\, dx = \int_{-\infty}^{\infty} \bar{F}(\mu)\, G(\mu)\, \frac{d\mu}{2\pi}\,,$$

$$\text{and} \qquad \int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(\mu)|^2\, \frac{d\mu}{2\pi}.$$

($\bar{}$ denotes complex conjugation).

5. Solve Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in the domain} \quad D = \{z: \; |z-1| > 1,\; |z-2| < 2\}$$

subject to the boundary condition

$$\phi(x,y) = a \text{ when } |z-1| = 1 \quad \text{and} \quad \phi(x,y) = b \text{ when } |z-2| = 2$$

($a$ and $b$ are real parameters).